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Separable systems with quadratic in momenta first integrals

Maciej Błaszak

Institute of Physics, A Mickiewicz University, Umultowska 85, 61-614 Poznań, Poland

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Abstract

The separability theory of Hamiltonian systems on Riemannian manifolds is reviewed and developed. Particular attention is paid to the systems generated by the so-called special conformal Killing tensors, i.e. Benenti systems. Then, infinitely many new classes of separable systems are constructed by appropriate deformations of Benenti class systems.

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1. Introduction

The separation of variables for solving by quadratures the Hamilton–Jacobi (HJ) equations of related Liouville integrable dynamic systems with quadratic in momenta first integrals has a long history as a part of analytical mechanics. There are some milestones of that theory. First, in 1891 Stäckel initiated a programme of classification of separable systems presenting conditions for separability of the HJ equations in orthogonal coordinates [1–3]. Then, in 1904 Levi-Civita found a test for the separability of a Hamiltonian dynamics in a given system of canonical coordinates [4]. The next was Eisenhart [5, 6], who in 1934 inserted the separability theory in the context of Riemannian geometry, making it coordinate free and introducing the crucial objects of the theory, i.e. the Killing tensors. This approach was then developed by Woodhouse [7], Klanins [8, 9] and others. Finally, in 1992, Benenti [10–12] constructed a particular but very important subclass of separable systems, based on the so-called special conformal Killing tensors.

The first constructive theory of separated coordinates for dynamic systems was proposed by Sklyanin [13]. He adopted the method of Lax representation for systematic derivation of separated coordinates. In that approach involutive functions appear as coefficients of the characteristic equation (spectral curve) of the Lax matrix. The method was successfully applied to separation of variables for many integrable systems [13–17].

Recently, a modern geometric theory of separability on bi-Poisson manifolds has been constructed [18–25]. Obviously, it contains as a special case the Liouville integrable systems with all constants of motion being quadratic in momenta functions.

In this paper we construct in a systematic way a separability theory of a large class of Liouville integrable systems on Riemannian manifolds, including as a special case the Benenti class of systems. More importantly, infinitely many new classes of separable systems are constructed from appropriate deformations of the Benenti class of systems. In that sense we demonstrate the crucial role of this particular class in the separability theory of dynamic systems on Riemannian manifolds.

The organization of the paper is as follows. In section 2 we sketch the separability theory of Hamiltonian systems on ωN manifolds, which has been recently constructed. Section 3 deals with a special case of separable systems, i.e. the so-called Benenti systems. We re-examine this class of systems systematically as it plays a crucial role in a separability theory on Riemannian manifolds and is of special importance for the theory developed in this paper. In section 4, we construct the simplest new classes of separable systems being the so-called one-hole deformations of the Benenti class. In this example we explain the main ideas of our approach as well as the methods of systematic construction of separable potentials and quasi-bi-Hamiltonian representations. In section 5, we develop the approach to the case of arbitrary k-hole deformations of the Benenti class of systems, constructing a complete theory of separable systems on Riemannian manifolds with separation conditions of polynomial type. Finally, in section 6, we illustrate our theory by a few simple examples.

2. Separable systems on ωN manifolds

Given a manifold $\mathcal{M}$ of $\dim \mathcal{M} = m$, a *Poisson operator* π on $\mathcal{M}$ is a bivector $\pi \in \Lambda^2(\mathcal{M})$ with vanishing Schouten bracket:

$$[\pi, \pi]_S = 0. \tag{2.1}$$

In a local coordinate system $(x^1, \ldots, x^m)$ we have

$$\pi = \sum_{i<j}^{m} \pi^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}, \tag{2.2}$$

while the Poisson property (2.1) takes the form

$$\sum_l (\pi^{jl}\partial_l \pi^{ik} + \pi^{il}\partial_l \pi^{kj} + \pi^{kl}\partial_l \pi^{ji}) = 0, \qquad \partial_i := \frac{\partial}{\partial x^i}. \tag{2.3}$$

Poisson tensor π, considered as the mapping $\pi : T^*\mathcal{M} \to T\mathcal{M}$, induces a bracket on the space $C^\infty(\mathcal{M})$ of all smooth real-valued functions on $\mathcal{M}$

$$\{.\,,.\}_\pi : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}), \qquad \{F, G\}_\pi \stackrel{\text{def}}{=} \langle \mathrm{d}F, \pi\, \mathrm{d}G\rangle = \pi(\mathrm{d}F, \mathrm{d}G), \tag{2.4}$$

(where $\langle .\,,.\rangle$ is the dual map between $T\mathcal{M}$ and $T^*\mathcal{M}$) which is skew-symmetric and satisfies the Jacobi identity. It is called a *Poisson bracket*.

A linear combination $\pi_\xi = \pi_1 + \xi\pi_0$ ($\xi \in \mathbb{R}$) of two Poisson operators π_0 and π_1 is called a *Poisson pencil* if the operator π_ξ is Poisson for any value of the parameter ξ, i.e. when $[\pi_0, \pi_1]_S = 0$. In this case we say that π_0 and π_1 are *compatible*.

Assume that π_ξ is a Poisson pencil on $\mathcal{M}$ and that Poisson tensor π_0 is nondegenerate. Hence, $\mathcal{M}$ is endowed with two 2-forms ω_0, ω_1 [27] defined by

$$\{F, G\}_{\pi_i} = \omega_i(X_F, X_G), \qquad X_F = \pi_0 \mathrm{d}F, \qquad i = 0, 1. \tag{2.5}$$

From (2.5) it follows that $\omega_0 = \pi_0^{-1}, \omega_1 = \omega_0\pi_1\omega_0$ and thus ω_0 is closed. Moreover, one can construct the tensor field $N := \pi_1\pi_0^{-1} = \pi_1\omega_0$, of the type $(1, 1)$, called a *recursion operator* of $\mathcal{M}$ and its dual $N^* = \omega_0\pi_1$. Note that

$$N\pi_0 = \pi_1, \qquad N^*\omega_0 = \omega_1. \tag{2.6}$$

The important property of N is that its Nijenhuis torsion vanishes as a consequence of the compatibility between π_0 and π_1 and hence implies that ω_1 is also closed [28]. Such manifolds are known as the so-called ωN manifolds. The generic case means that $2n$-dimensional ωN manifold is endowed with a recursion operator N which has at every point n distinct double eigenvalues $\lambda_1, \ldots, \lambda_n$, which are functionally independent of $\mathcal{M}$. Choosing λ_i as the canonical position coordinates, we can always supplement a set of local coordinates (λ_i, μ_i) on $\mathcal{M}$ by the canonically conjugate momenta μ_i.

Definition 1. *A set of local coordinates (λ_i, μ_i) on ωN manifold $\mathcal{M}$ is called a set of Darboux–Nijenhuis (DN) coordinates if they are canonical with respect to π_0 and diagonalize the recursion operator, whose diagonal elements are its eigenvalues.*

It means that in the (λ, μ) coordinates

$$\pi_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \qquad \pi_1 = \begin{pmatrix} 0 & \Lambda_n \\ -\Lambda_n & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} \Lambda_n & 0 \\ 0 & \Lambda_n \end{pmatrix}, \tag{2.7}$$

where $\Lambda_n = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, and their differentials span an eigenspace of N^* (the adjoint of N), as

$$N^* \mathrm{d}\lambda_i = \lambda_i \,\mathrm{d}\lambda_i, \qquad N^* \,\mathrm{d}\mu_i = \lambda_i \,\mathrm{d}\mu_i, \qquad i = 1, \ldots, n. \tag{2.8}$$

As is well known, n functionally independent Hamiltonian functions $H_i, i = 1, \ldots, n$ are said to be separable in the canonical coordinates (λ, μ) if there are n relations, called the separation conditions (Sklyanin [13]), of the form

$$\varphi_i(\lambda^i, \mu_i; H_1, \ldots, H_n) = 0, \qquad i = 1, \ldots, n, \qquad \det\left[\frac{\partial \varphi_i}{\partial H_j}\right] \neq 0, \tag{2.9}$$

which guarantee the solvability of the appropriate Hamilton–Jacobi equations and involutivity of H_i. An important special case, when all separation relations (2.9) are affine in H_i, is given by the set of equations

$$\sum_{k=1}^{n} \phi_i^k(\lambda_i, \mu_i) H_k = \psi_i(\lambda_i, \mu_i), \qquad i = 1, \ldots, n, \tag{2.10}$$

where ϕ_i^k and ψ_i are arbitrary smooth functions of their arguments. Equations (2.10) are called the *Stäckel separation conditions* and the related dynamic systems are called the Stäckel separable ones.

Theorem 2 [25]. *Let $\mathcal{M}$ be a generic ωN manifold and let $(H_1, \ldots, H_n)$ be a set of n functionally independent Hamiltonians on $\mathcal{M}$, separable in DN coordinates. Then, the subspace spanned by $(\mathrm{d}H_1, \ldots, \mathrm{d}H_n)$ is invariant with respect to N^*, i.e. there exist some functions α_{ij} such that*

$$N^* \,\mathrm{d}H_i = \sum_{j=1}^{n} \alpha_{ij} \,\mathrm{d}H_j, \qquad i = 1, \ldots, n. \tag{2.11}$$

Hence, the distribution defined by $(H_1, \ldots, H_n)$, spanned by the Hamiltonian vector fields X_{H_i}, is invariant with respect to N.

Formula (2.11) can be written in the equivalent form

$$\pi_1 \,\mathrm{d}H_i = \sum_{j=1}^{n} \alpha_{ij} \pi_0 \,\mathrm{d}H_j, \qquad i = 1, \ldots, n, \tag{2.12}$$

which will be called a *quasi-bi-Hamiltonian representation* of separable dynamics.

Theorem 3 [25]**.** *An n-tuple $(H_1, \ldots, H_n)$ of separable functions on ωN manifold $\mathcal{M}$ is Stäckel separable iff additionally to the condition (2.11) we have*

$$N^* \mathrm{d}\alpha_{ij} = \sum_{k=1}^{n} \alpha_{ik}\, \mathrm{d}\alpha_{kj}, \qquad i, j = 1, \ldots, n. \tag{2.13}$$

For the majority of known Stäckel integrable systems $\psi_i(\lambda_i, \mu_i) = \psi(\lambda_i, \mu_i)$ and $\phi_i^k(\lambda_i, \mu_i) = \phi^k(\lambda_i, \mu_i)$, and then, the separation conditions (2.10) can be presented in a compact form as n copies of the so-called *separation curve*

$$\sum_{k=1}^{n} \phi^k(\xi, \mu) H_k = \psi(\xi, \mu), \qquad (\xi, \mu) = (\lambda_i, \mu_i), \qquad i = 1, \ldots, n. \tag{2.14}$$

For the uniqueness of the representation (2.14) we have chosen the normalization condition $\phi^n(\xi, \mu) = 1$.

In this paper we restrict to a special but important class of systems where the function $\psi(\xi, \mu)$ is quadratic in momenta μ and multipliers $\phi^k(\xi, \mu)$ are monomials with respect to ξ

$$H_1\xi^{m_1} + \cdots + H_n\xi^{m_n} = \tfrac{1}{2} f(\xi)\mu^2 + \gamma(\xi), \qquad m_n = 0 < m_{n-1} < \cdots < m_1 \in \mathbb{N}, \tag{2.15}$$

where $f(\xi)$, and $\gamma(\xi)$ are Laurent polynomials of ξ. Separable systems from this class are dynamic systems on Riemannian manifolds.

Denoting by E_i the Hamiltonians of the geodesic case, i.e. the case with $\gamma(\xi) = 0$, and solving the system of separation conditions

$$E_1\xi^{m_1} + \cdots + E_n\xi^{m_n} = \tfrac{1}{2} f(\xi)\mu^2, \qquad (\xi, \mu) = (\lambda_i, \mu_i), \qquad i = 1, \ldots, n$$

with respect to E_i, one gets the original Stäckel representation [1–3]

$$E_r = \frac{1}{2} \sum_{i=1}^{n} (\varphi^{-1})_r^i \mu_i^2, \tag{2.16}$$

for separable geodesic Hamiltonians, where a nonsingular matrix $\varphi = \left(\varphi_k^l(\lambda_k)\right)$ is called the *Stäckel matrix*.

Eisenhart [5, 6] gave a coordinate-free representation for Stäckel geodesic motion introducing a special family of *Killing tensors*. Let (Q, g) be a Riemannian (pseudo-Riemannian) manifold with covariant metric tensor g and local coordinates $q^1, \ldots, q^n$. Moreover, let $G := g^{-1}$ be the contravariant metric tensor satisfying $\sum_{j=1}^{n} g_{ij} G^{jk} = \delta_i^k$. As known, a $(2, 0)$-type tensor $A = (A^{ij})$ is called a Killing tensor with respect to G if $\left\{\sum A^{ij} p_i p_j, E\right\}_{\pi_0} = 0$. Eisenhart proved [5, 6] that the geodesic Hamiltonians can be transformed into the Stäckel form (2.16) if the contravariant metric tensor G has $(n - 1)$ independent commuting contravariant Killing tensors A_r of second order such that

$$E_r = \frac{1}{2} p^T A_r p \equiv \frac{1}{2} \sum_{i,j} A_r^{ij} p_i p_j = \frac{1}{2} \sum_{i,j=1}^{n} (K_r G)^{ij} p_i p_j, \tag{2.17}$$

where $K_r = A_r g$ are $(1, 1)$-type Killing tensors.

From now on, separated canonical coordinates will be denoted by (λ, μ) and natural canonical coordinates by (q, p). For n degrees of freedom n Stäckel Hamiltonian functions are given in the form

$$H_r = \tfrac{1}{2} p^T K_r G p + V_r(q), \qquad r = 1, \ldots, n, \tag{2.18}$$

where $V_r(q)$ are appropriate separable potentials.

3. Systems of Benenti type

3.1. Separable geodesics

Among all Stäckel systems a particularly important subclass consists of these considered by Benenti [10–12] and constructed with the help of the so-called *conformal Killing tensor*. Let $L = \left(L^i_j\right)$ be a second-order mixed-type tensor on Q and let $\overline{L} : M \to \mathbb{R}$ be a function on M defined as $\overline{L} = \frac{1}{2}\sum_{i,j=1}^n (LG)^{ij} p_i p_j$. If

$$\{\overline{L}, E\}_{\pi_0} = \kappa E, \qquad \text{where} \quad \kappa = \{\varepsilon, E\}_{\pi_0}, \quad \varepsilon = \mathrm{Tr}(L), \tag{3.1}$$

then L is called a conformal Killing tensor with the associated potential $\varepsilon = \mathrm{Tr}(L)$. If we assume additionally that L has simple eigenvalues and its Nijenhuis torsion vanishes, then L is called a *special conformal Killing tensor* [29].

For the Riemannian manifold (Q, g, L), geodesic flow has n constants of motion of form (2.17), where Killing tensors K_r are constructed from L by the recursion formula

$$K_{r+1} = LK_r + \rho_r I, \qquad K_1 = I, \tag{3.2}$$

and ρ_r are coefficients of the characteristic polynomial of L

$$\det(\xi I - L) = \xi^n + \rho_1 \xi^{n-1} + \cdots + \rho_n, \qquad \rho_0 = 1. \tag{3.3}$$

Hence, functions E_r constitute a system of n constants of motion in involution with respect to the Poisson structure π_0. So, for a given metric tensor g, the existence of a special conformal Killing tensor L is a sufficient condition for the geodesic flow on Q to be a Liouville integrable Hamiltonian system with all constants of motion quadratic in momenta.

It turns out that with the tensor L we can (generically) associate a coordinate system on Q in which the geodesic flows associated with all the functions E_r separate. Namely, let $(\lambda^1(q), \ldots, \lambda^n(q))$ be n distinct, functionally independent eigenvalues of L, i.e. solutions of the characteristic equation $\det(\xi I - L) = 0$. Solving these relations with respect to q we get the transformation $q^i = \alpha_i(\lambda)$, $i = 1, \ldots, n$. The remaining part of the transformation to the separation coordinates can be obtained as a canonical transformation reconstructed from the generating function $W(p, \lambda) = \sum_i p_i \alpha_i(\lambda)$ in the standard way.

In the (λ, μ) coordinates the tensor L is diagonal $L = \mathrm{diag}(\lambda^1, \ldots, \lambda^n) \equiv \Lambda_n$, the geodesic Hamiltonians have the following form [18]:

$$E_r = -\frac{1}{2}\sum_{i=1}^n \frac{\partial \rho_r}{\partial \lambda^i} \frac{f_i(\lambda^i)}{\Delta_i} \mu_i^2, \qquad \Delta_i = \prod_{k=1,\ldots,n,k\neq i} (\lambda^i - \lambda^k), \tag{3.4}$$

where $\rho_r(\lambda)$ are symmetric polynomials (Viéte polynomials) defined by (3.3) and f_i are arbitrary smooth functions of one real argument. From (3.4) it immediately follows that in (λ, μ) variables the contravariant metric tensor G and all the Killing tensors K_r are diagonal

$$G^{ij} = \frac{f_i(\lambda^i)}{\Delta_i}\delta^{ij}, \qquad (K_r)^i_j = -\frac{\partial \rho_r}{\partial \lambda^i}\delta^i_j. \tag{3.5}$$

Remark 4. When $f_i(\lambda^i)$ is a polynomial of order $\leqslant n$ the Riemannian curvature tensor vanishes and the metric is flat, if the order of f is equal $n + 1$ the metric is of constant Riemannian curvature.

Separation conditions related to Hamiltonian functions (3.4) are as follows

$$E_1(\lambda^i)^{n-1} + E_2(\lambda^i)^{n-2} + \cdots + E_n = \tfrac{1}{2} f_i(\lambda^i)\mu_i^2, \qquad i = 1, \ldots, n. \tag{3.6}$$

For $f_i(\lambda^i) = f(\lambda^i)$ equations (3.6) can be represented by n different copies $(\xi, \mu) = (\lambda^i, \mu_i)$, $i = 1, \dots, n$ of some curve

$$E_1\xi^{n-1} + E_2\xi^{n-2} + \cdots + E_n = \tfrac{1}{2} f(\xi)\mu^2 \tag{3.7}$$

called the separation curve of geodesic motion for Benenti class systems.

3.2. Quasi-bi-Hamiltonian chains

The special conformal Killing tensor L can be lifted from Q to a (1, 1)-type tensor on $\mathcal{M} = T^*Q$ where it takes the form

$$N = \begin{pmatrix} L & 0 \\ F & L^* \end{pmatrix}, \qquad F^i_j = \frac{\partial}{\partial q^j}(p^T L)_i - \frac{\partial}{\partial q^i}(p^T L)_j, \qquad L^* = L^T. \tag{3.8}$$

The lifted (1, 1) tensor N is Niejhuis torsion free, like the L one, and is called a recursion operator on $\mathcal{M}$. An important property of N is that when it acts on the canonical Poisson tensor π_0 it produces another Poisson tensor

$$\pi_1 = N\pi_0 = \begin{pmatrix} 0 & L \\ -L^* & F \end{pmatrix}, \tag{3.9}$$

compatible with the canonical one (actually π_0 is compatible with $N^k\pi_0$ for any integer k) and $\mathcal{M}$ is the ωN manifold. It is now possible to show that the geodesic Hamiltonians E_r satisfy on $\mathcal{M} = T^*Q$ the set of relations [26]

$$\begin{gathered} \pi_1\, \mathrm{d}E_r = \pi_0\, \mathrm{d}E_{r+1} - \rho_r \pi_0\, \mathrm{d}E_1, \qquad E_{n+1} = 0, \qquad r = 1, \dots, n. \\ \Updownarrow \\ N^*\, \mathrm{d}E_r = \mathrm{d}E_{r+1} - \rho_r\, \mathrm{d}E_1, \qquad N^* = \pi_0^{-1}\pi_1 = \begin{pmatrix} L^* & -F \\ 0 & L \end{pmatrix}, \end{gathered} \tag{3.10}$$

which are a particular case of the quasi-bi-Hamiltonian chain (2.12) [30]. Note that the recursion relation (3.2) is immediately reconstructed from (3.10). Moreover, it is not difficult to show that L is the special conformal Killing tensor for each metric $G^{(s)} \equiv L^s G$.

Let us denote by $G^{(0)}$ the flat contravariant metric, which in λ coordinates takes the form

$$(G^{(0)})^{ij} = \frac{1}{\Delta_i}\delta^{ij}, \tag{3.11}$$

i.e. $f_i(\lambda^i) = 1$, $i = 1, \dots, n$ (3.5). It means that in the appropriate separation curve for geodesic motion (3.7) $f(\xi) = 1$. Moreover, the metric tensor G, which generates the separation curve (3.7) with $f(\xi)$ in the form of Laurent polynomial, is constructed from the metric $G^{(0)}$ in the following way: $G = f(L)G^{(0)}$.

3.3. Separable potentials

What potentials can be added to geodesic Hamiltonians E_r without destroying their separability within the above schema? It turns out that there exists a sequence of separable potentials $V_r^{(k)}$, $k = \pm 1, \pm 2, \dots$, which can be added to the geodesic Hamiltonians E_r such that the new Hamiltonians

$$H_r(q, p) = E_r(q, p) + V_r^{(k)}(q), \qquad r = 1, \dots, n, \tag{3.12}$$

are in involution with respect to π_0 and π_1 and are still separable in the same coordinates (λ, μ). It means that H_r follow the quasi-bi-Hamiltonian chain (3.10)

$$N^*\, \mathrm{d}H_r = \mathrm{d}H_{r+1} - \rho_r\, \mathrm{d}H_1, \qquad H_{n+1} = 0, \qquad r = 1, \dots, n, \tag{3.13}$$

while for potentials we have

$$L^* \, \mathrm{d}V_r = \mathrm{d}V_{r+1} - \rho_r \, \mathrm{d}V_1, \qquad r = 1, \dots, n. \tag{3.14}$$

Theorem 5. *Potentials $V_r^{(m)}$ given by the following recursion relation [19, 23, 31]:*

$$V_r^{(m+1)} = V_{r+1}^{(m)} - \rho_r V_1^{(m)}, \tag{3.15}$$

and its inverse

$$V_r^{(-m-1)} = V_{r-1}^{(-m)} - \frac{\rho_{r-1}}{\rho_n} V_n^{(-m)}, \qquad V_r^{(0)} = -\delta_{r,n} \tag{3.16}$$

are separable potentials.

Note that $V_r^{(m)} = -\delta_{r,n-m}$, $m = 0, \dots, n-1$, $V_r^{(n)} = \rho_r$, $V_r^{(-1)} = \frac{\rho_{r-1}}{\rho_n}$. Such notation will be useful in the case of deformed Benenti systems.

Lemma 6. *Nontrivial potentials $V_r^{(n-1+k)}$ and $V_r^{(-k)}$, $k = 1, 2, \dots$ added to the geodesic Hamiltonians E_i, $i = 1, \dots, n$ transform the separation curve (3.7) to the form*

$$H_1 \xi^{n-1} + H_2 \xi^{n-2} + \cdots + H_n = \tfrac{1}{2} f(\xi) \mu^2 + \gamma(\xi), \tag{3.17}$$

where $\gamma(\xi) = -\xi^{n-1+k}$, for potentials $V^{(n-1+k)}$ and $\gamma(\xi) = -\xi^{-k}$ for potentials $V^{(-k)}$, respectively.

Proof. Potentials $V_r^{(n)} = \rho_r$ are coefficients of characteristic equation of the special conformal Killing tensor L

$$\xi^n + \rho_1 \xi^{n-1} + \cdots + \rho_n = 0. \tag{3.18}$$

Then, we define $V^{(n+k)}$ potentials by a generating equation

$$\xi^{n+k} + V_1^{(n+k)} \xi^{n-1} + \cdots + V_n^{(n+k)} = 0. \tag{3.19}$$

Hence, adding equations (3.7) and (3.19), we get the separation curve (3.17). On the other hand, recursion formula (3.15) is reconstructed as follows. From (3.19) we have

$$\xi^{n+k+1} + V_1^{(n+k)} \xi^n + \cdots + V_n^{(n+k)} \xi = 0. \tag{3.20}$$

Elimination of ξ^n via (3.18) leads to the form

$$\begin{gathered} \xi^{n+k+1} + \big(V_2^{(n+k)} - \rho_1 V_1^{(n+k)}\big)\xi^{n-1} + \cdots + \big(V_n^{(n+k)} - \rho_{n-1} V_1^{(n+k)}\big)\xi - \rho_n V_1^{(n+k)} = 0 \\ \Downarrow \\ V_r^{(n+k+1)} = V_{r+1}^{(n+k)} - \rho_r V_1^{(n+k)}. \end{gathered} \tag{3.21}$$

For the inverse potentials the proof is similar. □

4. One-hole deformation of Benenti systems

Separable systems on Riemannian manifolds considered by Benenti belong to an important but very particular subclass of such systems. In this context, a question arises about the classification of all separable systems on Riemannian manifolds, with n quadratic in momenta constants of motion. The classification can be made with respect to the admissible forms of Stäckel separability conditions. The right-hand side of conditions (2.15) is always the same for the class of systems considered

$$\mathrm{rhs} = \tfrac{1}{2} f_i(\lambda^i) \mu_i^2 + \gamma_i(\lambda^i) = \psi(\lambda^i, \mu_i), \tag{4.1}$$

so different classes of separable systems are described by different forms of the lhs of Stäckel conditions. In the simplest Benenti case, it is given by the following polynomial form:

$$\text{lhs} = H_1\xi^{n-1} + H_2\xi^{n-2} + \cdots + H_n, \qquad \xi = \lambda^1, \ldots, \lambda^n. \tag{4.2}$$

We will show that all other classes, given by some polynomial in λ, are appropriate deformations of the Benenti class.

First, let us define a one-hole deformation of the Benenti class. For fixed n and n_1, where $1 < n_1 < n+1$, consider the following separability condition:

$$\widetilde{H}_1\xi^{(n+1)-1} + \widetilde{H}_2\xi^{(n+1)-2} + \cdots + \widetilde{H}_{n+1} = \psi(\xi, \mu), \qquad \widetilde{H}_{n_1} = 0 \tag{4.3}$$

and the Benenti separability condition with the same ψ representation

$$H_1\xi^{n-1} + H_2\xi^{n-2} + \cdots + H_n = \psi(\xi, \mu). \tag{4.4}$$

Note that all Benenti systems are classified by different forms of ψ, i.e. by $f_i(\lambda^i)$ and $\gamma_i(\lambda^i)$. A missing monomial (a hole) in (4.3) is $\widetilde{H}_{n_1}\xi^{(n+1)-n_1}$. Using the characteristic equation of a conformal Killing tensor L of the Benenti system (4.4)

$$\xi^n + \rho_1\xi^{n-1} + \cdots + \rho_n = 0,$$

for elimination of ξ^n, equation (4.3) can be transformed to the form

$$(\widetilde{H}_2 - \rho_1\widetilde{H}_1)\xi^{n-1} + \cdots + (\widetilde{H}_{n+1} - \rho_n\widetilde{H}_1) = \psi(\xi, \mu) \tag{4.5}$$

and hence, comparing it with (4.4) we find

$$H_r = \widetilde{H}_{r+1} - \rho_r\widetilde{H}_1, \qquad r = 1, \ldots, n, \tag{4.6}$$

with the inverse

$$\widetilde{H}_{r+1} = H_r - \frac{\rho_r}{\rho_{n_1-1}}H_{n_1-1}, \qquad r = 0, \ldots, n, \tag{4.7}$$

where $H_0 = 0$, $\rho_0 = 1$.

Note, that formula (4.7) applies separately to the geodesic and the potential parts, i.e.

$$\widetilde{E}_{r+1} = E_r - \frac{\rho_r}{\rho_{n_1-1}}E_{n_1-1}, \tag{4.8a}$$

$$\widetilde{V}_{r+1} = V_r - \frac{\rho_r}{\rho_{n_1-1}}V_{n_1-1}, \qquad r = 0, \ldots, n. \tag{4.8b}$$

4.1. Geodesic part

Let us first look at geodesic Hamiltonians

$$\widetilde{E}_r = \frac{1}{2}p^T\left(\rho_{n_1-1}K_{r-1} - \rho_{r-1}K_{n_1-1}\right)\frac{1}{\rho_{n_1-1}}Gp, \qquad r = 1, \ldots, n+1. \tag{4.9}$$

Using the known relation for the Benenti chain $\rho_r I = K_{r+1} - LK_r$ we get

$$\widetilde{E}_1 = -\frac{1}{\rho_{n_1-1}}E_{n_1-1} = -\frac{1}{\rho_{n_1-1}}\frac{1}{2}p^TK_{n_1-1}Gp = \frac{1}{2}p^T\widetilde{G}p \Longrightarrow \widetilde{G} = -\frac{1}{\rho_{n_1-1}}K_{n_1-1}G \tag{4.10}$$

and

$$\widetilde{E}_r = \frac{1}{2}p^T\left[\frac{1}{\rho_{n_1-1}}\left(K_{n_1}K_{r-1} - K_{n_1-1}K_r\right)G\right]p = \frac{1}{2}p^T\widetilde{K}_r\widetilde{G}p$$

$$\Longrightarrow \widetilde{K}_r = K_r - K_{r-1}K_{n_1}\left(K_{n_1-1}\right)^{-1}. \tag{4.11}$$

The $\widetilde{E}_r$ functions are in involution because they fulfil Stäckel separation conditions (2.10).

4.2. Deformed potentials

Let us analyse the basic deformed potentials. The first potentials are the following. From (4.8*b*) and the Benenti potentials we have $\widetilde{V}_r^{(m)} = -\delta_{r-1,n-m}$ for $m < n+1$, $m \neq (n+1) - n_1$ and

$$\widetilde{V}_r^{(n+1)-n_1} = \frac{\rho_{r-1}}{\rho_{n_1-1}}, \qquad \widetilde{V}_r^{(n+1)} = \rho_r - \frac{\rho_{r-1}\rho_{n_1}}{\rho_{n_1-1}}, \ldots. \tag{4.12}$$

Note that $\widetilde{V}_{n_1}^{(m)} = 0$ for $m \geqslant n+1$ and $\widetilde{V}_{n_1}^{(n+1)-n_1} = 1$.

Lemma 7. *Nontrivial basic potentials* $\widetilde{V}_r^{(n+1-n_1)}, \widetilde{V}_r^{(n+k)}$ *and* $\widetilde{V}_r^{(-k)}, k = 1, 2, \ldots$ *enter the separation curve*

$$\widetilde{H}_1\xi^n + \widetilde{H}_2\xi^{n-1} + \cdots + \widetilde{H}_{n+1} = \tfrac{1}{2}f(\xi)\mu^2 + \gamma(\xi), \qquad \widetilde{H}_{n_1} = 0, \tag{4.13}$$

as $\gamma(\xi) = -\xi^{(n+1)-n_1}, -\xi^{n+k}, -\xi^{-k}$.

Proof. We will show the following generating equations for the potentials considered:

$$\xi^{n+k} + \widetilde{V}_1^{(n+k)}\xi^n + \cdots + 0\xi^{n+1-n_1} + \cdots + \widetilde{V}_{n+1}^{(n+k)} = 0, \qquad k = 1, 2, \ldots, \tag{4.14a}$$

$$\xi^{-k} + \widetilde{V}_1^{(-k)}\xi^n + \cdots + 0\xi^{n+1-n_1} + \cdots + \widetilde{V}_{n+1}^{(-k)} = 0, \qquad k = 1, 2, \ldots, \tag{4.14b}$$

$$\widetilde{V}_1^{(n+1-n_1)}\xi^n + \cdots + \xi^{n+1-n_1} + \cdots + \widetilde{V}_{n+1}^{(n+1-n_1)} = 0. \tag{4.14c}$$

For the first two equations we have ($m > n$ or $m < 0$)

$$\begin{gathered}
\xi^m + \widetilde{V}_1^{(m)}\xi^n + \cdots + \widetilde{V}_{n+1}^{(m)} = 0, \qquad \widetilde{V}_{n_1}^{(m)} = 0 \\
\Updownarrow \\
\xi^m + \widetilde{V}_1^{(m)}(-\rho_1\xi^{n-1} - \cdots - \rho_n) + \cdots + \widetilde{V}_{n+1}^{(m)} = 0 \\
\Updownarrow \\
\xi^m + \big(\widetilde{V}_2^{(m)} - \rho_1\widetilde{V}_1^{(m)}\big)\xi^{n-1} + \cdots + \big(\widetilde{V}_{n+1}^{(m)} - \rho_n\widetilde{V}_1^{(m)}\big) = 0 \\
\Updownarrow \\
\xi^m + V_1^{(m)}\xi^{n-1} + \cdots + V_n^{(m)} = 0
\end{gathered}$$

which reveal the known deformation relations (4.8*b*)

$$V_r^{(m)} = \widetilde{V}_{r+1}^{(m)} - \rho_r\widetilde{V}_1^{(m)} \Longleftrightarrow \widetilde{V}_{r+1}^{(m)} = V_r^{(m)} - \frac{\rho_r}{\rho_{n_1-1}}V_{n_1-1}^{(m)} \tag{4.15}$$

between nontrivial basic potentials from Benenti class and respective deformed potentials. For the last case (4.14*c*) we have

$$\begin{gathered}
\widetilde{V}_1^{(n+1-n_1)}\xi^n + \cdots + \xi^{n+1-n_1} + \cdots + \widetilde{V}_{n+1}^{(n+1-n_1)} = 0, \\
\Updownarrow \\
\xi^n + \frac{\widetilde{V}_2^{(n+1-n_1)}}{\widetilde{V}_1^{(n+1-n_1)}}\xi^{n-1} + \cdots + \frac{1}{\widetilde{V}_1^{(n+1-n_1)}}\xi^{n+1-n_1} + \cdots + \frac{\widetilde{V}_{n+1}^{(n+1-n_1)}}{\widetilde{V}_1^{(n+1-n_1)}} = 0 \\
\Updownarrow \\
\xi^n + \rho_1\xi^{n-1} + \cdots + \rho_n = 0 \Longrightarrow \widetilde{V}_r^{(n+1)-n_1} = \frac{\rho_{r-1}}{\rho_{n_1-1}}
\end{gathered}$$

which is a special case of deformations (4.15) related to the trivial Benenti potential $V_r^{(n+1-n_1)} = -\delta_{r,n_1-1}$. □

Alternatively, the basic potentials $\widetilde{V}^{(m)}$, $m > n+1$ can be constructed recursively as in the Benenti case.

Lemma 8. *The basic separable potentials* $\widetilde{V}_r^{(m)}$, $m > n+1$ *are given by the following recursion relation:*

$$\widetilde{V}_r^{(m+1)} = \widetilde{V}_{r+1}^{(m)} - \widetilde{V}_r^{(n+1)}\widetilde{V}_1^{(m)} - \widetilde{V}_r^{(n+1-n_1)}\widetilde{V}_{n_1+1}^{(m)}, \tag{4.16}$$

where $\widetilde{V}_r^{(n+1)-n_1}$ *and* $\widetilde{V}_r^{(n+1)}$ *are given by (4.12).*

Proof. The potentials $\widetilde{V}_r^{(n+1)-n_1}$ enter the separation curve in the form (4.14*c*), while the potentials $\widetilde{V}_r^{(n+1)}, \widetilde{V}_r^{(m)}, \widetilde{V}_r^{(m+1)}$ enter the separation curve in the form (4.14*a*) with $k = 1, m-n, m+1-n$. The recursion formula (4.16) is reconstructed as follows. Multiplying equation (4.14*a*) for $k = m - n$ by ξ we have

$$\xi^{m+1} + \widetilde{V}_1^{(m)}\xi^{n+1} + \cdots + \widetilde{V}_{n_1+1}^{(m)}\xi^{n+1-n_1} + \cdots + \widetilde{V}_{n+1}^{(m)}\xi = 0.$$

Substituting ξ^{n+1} from (4.14*a*) for $k = 1$ and ξ^{n+1-n_1} from (4.14*c*) we get

$$\begin{aligned}\xi^{m+1} &+ \widetilde{V}_1^{(m)}\big(-\widetilde{V}_1^{(n+1)}\xi^n - \cdots - \widetilde{V}_{n+1}^{(n+1)}\big) + \cdots \\ &+ \widetilde{V}_{n_1+1}^{(m)}\big(-\widetilde{V}_1^{(n+1-n_1)}\xi^n - \cdots - \widetilde{V}_{n+1}^{(n+1-n_1)}\big) + \cdots + \widetilde{V}_{n+1}^{(m)}\xi = 0.\end{aligned}$$

A comparison with the separation curve for the potential $\widetilde{V}_r^{(m+1)}$ (equation (4.14*a*) with $k = m + 1 - n$) reveals formula (4.16). □

Of course, from the construction, all Hamiltonian functions $\widetilde{H}_r$ are in involution with respect to π_0.

4.3. *Quasi-bi-Hamiltonian representation*

Theorem 9. *Hamiltonian functions* $\widetilde{H}_r$ *fulfil the following quasi-bi-Hamiltonian chain:*

$$\begin{gathered}\mathrm{d}\widetilde{H}_{r+1} = N^*\,\mathrm{d}\widetilde{H}_r + \alpha_r\,\mathrm{d}\widetilde{H}_1 + \beta_r\,\mathrm{d}\widetilde{H}_{n_1+1} \\ \Updownarrow \\ \pi_0\,\mathrm{d}\widetilde{H}_{r+1} = \pi_1\,\mathrm{d}\widetilde{H}_r + \alpha_r\pi_0\,\mathrm{d}\widetilde{H}_1 + \beta_r\pi_0\,\mathrm{d}\widetilde{H}_{n_1+1},\end{gathered} \tag{4.17}$$

where $\alpha_r = \widetilde{V}_r^{(n+1)}$, $\beta_r = \widetilde{V}_r^{(n+1)-n_1}$.

Proof. We use the property of the Benenti chain

$$\mathrm{d}H_{r+1} = N^*\,\mathrm{d}H_r + V_r^{(n)}\,\mathrm{d}H_1, \tag{4.18a}$$

$$\mathrm{d}\rho_{r+1} = L^*\,\mathrm{d}\rho_r + V_r^{(n)}\,\mathrm{d}\rho_1, \qquad V_r^{(n)} = \rho_r, \tag{4.18b}$$

and recursion relations (4.7), (4.8*b*). Hence, we have

$$\begin{aligned}\mathrm{rhs}(4.17) &= N^*\,\mathrm{d}\widetilde{H}_r + \alpha_r\,\mathrm{d}\widetilde{H}_1 + \beta_r\,\mathrm{d}\widetilde{H}_{n_1+1} \\ &= N^*\,\mathrm{d}\left(H_{r-1} - \frac{\rho_{r-1}}{\rho_{n_1-1}}H_{n_1-1}\right) + \left(\rho_r - \frac{\rho_{r-1}\rho_{n_1}}{\rho_{n_1-1}}\right)\mathrm{d}\left(-\frac{1}{\rho_{n_1-1}}H_{n_{1-1}}\right) \\ &\quad + \frac{\rho_{r-1}}{\rho_{n_1-1}}\left(H_{n_1} - \frac{\rho_{n_1}}{\rho_{n_1-1}}H_{n_1-1}\right) \\ &= N^*\,\mathrm{d}H_{r-1} - \frac{\rho_{r-1}}{\rho_{n_1-1}}N^*\,\mathrm{d}H_{n_1-1} - \rho_{r-1}H_{n_1-1}N^*\,\mathrm{d}\left(\frac{1}{\rho_{n_1-1}}\right) \\ &\quad - \frac{1}{\rho_{n_1-1}}H_{n_1-1}N^*\,\mathrm{d}\rho_{r-1} + \frac{\rho_{r-1}}{\rho_{n_1-1}}\,\mathrm{d}H_{n_1} - \frac{\rho_{r-1}}{\rho_{n_1-1}^2}H_{n_1-1}\,\mathrm{d}\rho_{n_1} - \rho_r\,\mathrm{d}\left(\frac{H_{n_1-1}}{\rho_{n_1-1}}\right) \\ &= \mathrm{d}H_r - \rho_r\,\mathrm{d}\left(\frac{H_{n_1-1}}{\rho_{n_1-1}}\right) - \frac{\rho_{r-1}}{\rho_{n_1-1}^2}H_{n_1-1}\,\mathrm{d}\rho_1 - \frac{1}{\rho_{n_1-1}}H_{n_1-1}N^*\,\mathrm{d}\rho_{r-1}\end{aligned}$$

$$= \mathrm{d}H_r - \rho_r\,\mathrm{d}\left(\frac{H_{n_1-1}}{\rho_{n_1-1}}\right) - \frac{H_{n_1-1}}{\rho_{n_1-1}}(N^*\,\mathrm{d}\rho_{r-1} + \rho_{r-1}\,\mathrm{d}\rho_1)$$

$$= \mathrm{d}H_r - \rho_r\,\mathrm{d}\left(\frac{H_{n_1-1}}{\rho_{n_1-1}}\right) - \frac{H_{n_1-1}}{\rho_{n_1-1}}\,\mathrm{d}\rho_r$$

$$= \mathrm{d}\left(H_r - \frac{\rho_r}{\rho_{n_1-1}}H_{n_1-1}\right) = \mathrm{d}\widetilde{H}_{r+1} = \mathrm{lhs}(4.17). \qquad \square$$

Of course, formula (4.17) works separately for $\widetilde{E}_r$ and $\widetilde{V}_r$ in the form

$$\mathrm{d}\widetilde{E}_{r+1} = N^*\,\mathrm{d}\widetilde{E}_r + \alpha_r\,\mathrm{d}\widetilde{E}_1 + \beta_r\,\mathrm{d}\widetilde{E}_{n_1+1}, \tag{4.19a}$$

$$\mathrm{d}\widetilde{V}_{r+1} = L^*\,\mathrm{d}\widetilde{V}_r + \alpha_r\,\mathrm{d}\widetilde{V}_1 + \beta_r\,\mathrm{d}\widetilde{V}_{n_1+1}. \tag{4.19b}$$

In analogy to the Benenti case, components $\mathrm{d}p_i$ of (4.19*a*) give us immediately the analogue of the recursion formula (3.2) for the one-hole case

$$\widetilde{K}_{r+1} = L\widetilde{K}_r + \alpha_r I + \beta_r \widetilde{K}_{n_1+1}. \tag{4.20}$$

Let us introduce the $\widetilde{L}$ function as was done for the Benenti case: $\widetilde{L} = \frac{1}{2}p^T L\widetilde{G}p$. Then, from (4.20) we find

$$\widetilde{L} = \widetilde{E}_2 - \alpha_1\widetilde{E}_1 - \beta_1\widetilde{E}_{n_1+1} \tag{4.21}$$

and

$$\{\widetilde{L}, \widetilde{E}_1\}_{\pi_0} = \{\widetilde{E}_1, \alpha_1\}_{\pi_0}\widetilde{E}_1 + \{\widetilde{E}_1, \beta_1\}_{\pi_0}\widetilde{E}_{n_1+1} = \kappa^0\widetilde{E}_1 + \kappa^{n_1}\widetilde{E}_{n_1+1}. \tag{4.22}$$

Thus, for $\widetilde{G}$, L is not a conformal Killing tensor.

5. *k*-hole deformations of Benenti systems

5.1. Deformation procedure

Here we extend the results of the previous section to the general k-hole case. Let us start with the separability condition

$$\widetilde{H}_1\xi^{(n+k)-1} + \widetilde{H}_2\xi^{(n+k)-2} + \cdots + \widetilde{H}_{n+k} = \psi(\xi, \mu), \tag{5.1}$$

with k-holes in $\xi^{(n+k)-n_1}, \xi^{(n+k)-n_2}, \ldots, \xi^{(n+k)-n_k}$, $1 < n_1 < \cdots < n_k < n+k$, $k \in \mathbb{N}$, i.e. $\widetilde{H}_{n_1} = \widetilde{H}_{n_2} = \cdots = \widetilde{H}_{n_k} = 0$, and the separability condition for Benenti systems with the same ψ

$$H_1\xi^{n-1} + H_2\xi^{n-2} + \cdots + H_n = \psi(\xi, \mu). \tag{5.2}$$

As for the basic potentials

$$\xi^{n+k} + V_1^{(n+k)}\xi^{n-1} + \cdots + V_n^{(n+k)} = 0,$$

substituting this relation into (5.1) for $\xi^{(n+k)-1}, \ldots, \xi^n$ we get a deformation of the chain (5.1) to the Benenti case (5.2)

$$H_r = \widetilde{H}_{r+k} - V_r^{(n+k-1)}\widetilde{H}_1 - V_r^{(n+k-2)}\widetilde{H}_2 - \cdots - V_r^{(n)}\widetilde{H}_k, \qquad r = 1, \ldots, n, \tag{5.3}$$

where $\widetilde{H}_{n_1} = \cdots = \widetilde{H}_{n_k} = 0$ and $V_r^{(m)}$ are appropriate basic Benenti potentials.

Lemma 10. *Deformation of the Benenti case (5.2) to the chain (5.1), i.e. the inverse formula to (5.3) one, is given by the following determinant form:*

$$\widetilde{H}_r = \frac{\begin{vmatrix} H_{r-k} & \rho_{r-1} & \cdots & \rho_{r-k} \\ H_{n_1-k} & \rho_{n_1-1} & \cdots & \rho_{n_1-k} \\ \cdots & \cdots & \cdots & \cdots \\ H_{n_k-k} & \rho_{n_k-1} & \cdots & \rho_{n_k-k} \end{vmatrix}}{\begin{vmatrix} \rho_{n_1-1} & \cdots & \rho_{n_1-k} \\ \cdots & \cdots & \cdots \\ \rho_{n_k-1} & \cdots & \rho_{n_k-k} \end{vmatrix}}. \tag{5.4}$$

Proof. First, we select from (5.3) k equations containing $\widetilde{H}_{n_1}, \ldots, \widetilde{H}_{n_k}$

$$\begin{aligned} H_{n_1-k} &= -V_{n_1-k}^{(n+k-1)} \widetilde{H}_1 - \cdots - V_{n_1-k}^{(n)} \widetilde{H}_k, \\ &\vdots \\ H_{n_k-k} &= -V_{n_k-k}^{(n+k-1)} \widetilde{H}_1 - \cdots - V_{n_k-k}^{(n)} \widetilde{H}_k. \end{aligned}$$

The solution with respect to $\widetilde{H}_i$, $i = 1, \ldots, k$ is given by a determinant form

$$\widetilde{H}_i = \frac{W_i}{W}, \qquad i = 1, \ldots, n,$$

where

$$W = (-1)^k \begin{vmatrix} V_{n_1-k}^{(n+k-1)} & \cdots & V_{n_1-k}^{(n)} \\ \cdots & \cdots & \cdots \\ V_{n_k-k}^{(n+k-1)} & \cdots & V_{n_k-k}^{(n)} \end{vmatrix}, \qquad W_i = (-1)^{k+i} \begin{vmatrix} H_{n_1-k} & V_{n_1-k}^{(n+k-1)} & \cdots & V_{n_1-k}^{(n)} \\ \cdots & \cdots & \cdots & \cdots \\ H_{n_k-k} & V_{n_k-k}^{(n+k-1)} & \cdots & V_{n_k-k}^{(n)} \end{vmatrix}$$

with the column $\left(V_{n_1-k}^{(n+k-i)}, \ldots, V_{n_k-k}^{(n+k-i)}\right)^T$ missing in W_i. Substituting this result to (5.3) we get

$$\begin{aligned} \widetilde{H}_r &= \frac{H_{r-k} W + V_{r-k}^{(n+k-1)} W_1 + \cdots + V_{r-k}^{(n)} W_k}{W} \\ &= \frac{\begin{vmatrix} H_{r-k} & V_{r-k}^{(n+k-1)} & \cdots & V_{r-k}^{(n)} \\ H_{n_1-k} & V_{n_1-k}^{(n+k-1)} & \cdots & V_{n_1-k}^{(n)} \\ \cdots & \cdots & \cdots & \cdots \\ H_{n_k-k} & V_{n_k-k}^{(n+k-1)} & \cdots & V_{n_k-k}^{(n)} \end{vmatrix}}{\begin{vmatrix} V_{n_1-k}^{(n+k-1)} & \cdots & V_{n_1-k}^{(n)} \\ \cdots & \cdots & \cdots \\ V_{n_k-k}^{(n+k-1)} & \cdots & V_{n_k-k}^{(n)} \end{vmatrix}} = \frac{\begin{vmatrix} H_{r-k} & \rho_{r-1} & \cdots & \rho_{r-k} \\ H_{n_1-k} & \rho_{n_1-1} & \cdots & \rho_{n_1-k} \\ \cdots & \cdots & \cdots & \cdots \\ H_{n_k-k} & \rho_{n_k-1} & \cdots & \rho_{n_k-k} \end{vmatrix}}{\begin{vmatrix} \rho_{n_1-1} & \cdots & \rho_{n_1-k} \\ \cdots & \cdots & \cdots \\ \rho_{n_k-1} & \cdots & \rho_{n_k-k} \end{vmatrix}}. \end{aligned}$$

The last step is valid due to the fact that $V_i^{(n)} = \rho_i$, the form of the recursion formula for Benenti basic potentials (3.15) and the properties of the determinants. It allows us to replace the arbitrary potential $V^{(n+k-i)}$ in the determinants by the $V^{(n)} = \rho$ one. For each recursive step we have

$$\begin{vmatrix} \cdots & V^{(n+k-i)}_{n_1-k} & \cdots & \rho_{n_1-k} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & V^{(n+k-i)}_{n_k-k} & \cdots & \rho_{n_k-k} \end{vmatrix} = \begin{vmatrix} \cdots & V^{(n+k-i-1)}_{n_1-k+1} - \rho_{n_1-k} V^{(n+k-i-1)}_1 & \cdots & \rho_{n_1-k} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & V^{(n+k-i-1)}_{n_k-k+1} - \rho_{n_k-k} V^{(n+k-i-1)}_1 & \cdots & \rho_{n_k-k} \end{vmatrix}$$

$$= \begin{vmatrix} \cdots & V^{(n+k-i-1)}_{n_1-k+1} & \cdots & \rho_{n_1-k} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & V^{(n+k-i-1)}_{n_k-k+1} & \cdots & \rho_{n_k-k} \end{vmatrix}. \qquad \square$$

Formula (5.4) applies separately to the geodesic and the potential parts.

5.2. Deformed geodesic motion

Let us first look at n geodesic Hamiltonians $\widetilde{E}_r$, $r = 1, \ldots, n+k$, $r \neq n_1, \ldots, n_k$. Then one finds

$$\widetilde{E}_r = \frac{1}{\varphi} \begin{vmatrix} E_{r-k} & \rho_{r-1} & \cdots & \rho_{r-k} \\ E_{n_1-k} & \rho_{n_1-1} & \cdots & \rho_{n_1-k} \\ \cdots & \cdots & \cdots & \cdots \\ E_{n_k-k} & \rho_{n_k-1} & \cdots & \rho_{n_k-k} \end{vmatrix}, \qquad \varphi(n_1, \ldots, n_k) = \begin{vmatrix} \rho_{n_1-1} & \cdots & \rho_{n_1-k} \\ \cdots & \cdots & \cdots \\ \rho_{n_k-1} & \cdots & \rho_{n_k-k} \end{vmatrix}. \tag{5.5}$$

Using the known relations for a Benenti chain $\rho_r I = K_{r+1} - L K_r$ and the property of determinants we get

$$\widetilde{E}_r = \frac{1}{2} p^T \begin{vmatrix} K_{r-k} & \rho_{r-1} & \cdots & \rho_{r-k} \\ K_{n_1-k} & \rho_{n_1-1} & \cdots & \rho_{n_1-k} \\ \cdots & \cdots & \cdots & \cdots \\ K_{n_k-k} & \rho_{n_k-1} & \cdots & \rho_{n_k-k} \end{vmatrix} \frac{1}{\varphi} G p = \frac{1}{2} p^T \begin{vmatrix} K_{r-k} & K_r & \cdots & K_{r-k+1} \\ K_{n_1-k} & K_{n_1} & \cdots & K_{n_1-k+1} \\ \cdots & \cdots & \cdots & \cdots \\ K_{n_k-k} & K_{n_k} & \cdots & K_{n_k-k+1} \end{vmatrix} \frac{1}{\varphi} G p$$

$$= (-1)^k \frac{1}{2} p^T \begin{vmatrix} K_r & \cdots & K_{r-k+1} & K_{r-k} \\ K_{n_1} & \cdots & K_{n_1-k+1} & K_{n_1-k} \\ \cdots & \cdots & \cdots & \cdots \\ K_{n_k} & \cdots & K_{n_k-k+1} & K_{n_k-k} \end{vmatrix} \frac{1}{\varphi} G p$$

$$= (-1)^k \frac{1}{2} p^T (K_r D_0 - K_{r-1} D_1 + \cdots + (-1)^k K_{r-k} D_k) \frac{1}{\varphi} G p, \tag{5.6}$$

where

$$D_i = \begin{vmatrix} K_{n_1} & \cdots & K_{n_1-i+1} & K_{n_1-i-1} & \cdots & K_{n_1-k+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ K_{n_k} & \cdots & K_{n_k-i+1} & K_{n_k-i-1} & \cdots & K_{n_k-k+1} \end{vmatrix}, \qquad i = 0, \ldots, k$$

and K_m in determinant calculations are treated as symbols not matrices. Then,

$$\widetilde{E}_1 = \frac{1}{2} p^T \widetilde{G} p \quad \Longrightarrow \quad \widetilde{G} = (-1)^k \frac{1}{\varphi} D_0 G, \tag{5.7}$$

and

$$\widetilde{E}_r = \tfrac{1}{2} p^T \widetilde{K}_r \widetilde{G} p \quad \Longrightarrow \quad \widetilde{K}_r = K_r - K_{r-1} D_1 D_0^{-1} + \cdots + (-1)^k K_{r-k} D_k D_0^{-1}, \tag{5.8}$$

where

$$D_0 = \begin{vmatrix} K_{n_1-1} & \cdots & K_{n_1-k} \\ \cdots & \cdots & \cdots \\ K_{n_k-1} & \cdots & K_{n_k-k} \end{vmatrix}. \tag{5.9}$$

Again we know from the construction that $\widetilde{E}_r$ are in involution, as they fulfil separation conditions (2.10).

5.3. Basic deformed potentials

From (5.4) the deformed potentials are

$$\widetilde{V}_r^{(m)} = \frac{1}{\varphi(n_1,\ldots,n_k)} \begin{vmatrix} V_{r-k}^{(m)} & \rho_{r-1} & \cdots & \rho_{r-k} \\ V_{n_1-k}^{(m)} & \rho_{n_1-1} & \cdots & \rho_{n_1-k} \\ \cdots & \cdots & \cdots & \cdots \\ V_{n_k-k}^{(m)} & \rho_{n_k-1} & \cdots & \rho_{n_k-k} \end{vmatrix}, \tag{5.10}$$

so using the recursion formula for the Benenti potentials and the properties of the determinants we have $\widetilde{V}_r^{(m)} = -\delta_{r-k,n-m}$, $m < n+k$, $m \neq (n+k) - n_i$, $i = 1,\ldots,k$,

$$\widetilde{V}_r^{(n+k)-n_i} = (-1)^{i+1}\frac{1}{\varphi} \begin{vmatrix} \rho_{r-1} & \cdots & \rho_{r-k} \\ \rho_{n_1-1} & \cdots & \rho_{n_1-k} \\ \cdots & \cdots & \cdots \\ \rho_{n_k-1} & \cdots & \rho_{n_k-k} \end{vmatrix} \tag{5.11}$$

with the row $(\rho_{n_i-1},\ldots,\rho_{n_i-k})$ missing,

$$\widetilde{V}_r^{(n+k)} = \frac{1}{\varphi} \begin{vmatrix} \rho_r & \rho_{r-1} & \cdots & \rho_{r-k} \\ \rho_{n_1} & \rho_{n_1-1} & \cdots & \rho_{n_1-k} \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{n_k} & \rho_{n_k-1} & \cdots & \rho_{n_k-k} \end{vmatrix}, \ldots. \tag{5.12}$$

Note that $\widetilde{V}_{n_i}^{(m)} = 0$ for arbitrary $m \geqslant n+k$ and $\widetilde{V}_{n_i}^{(n+k)-n_j} = \delta_{ij}$, $i, j = 1,\ldots,k$.

As in the one-hole case, one can show that the nontrivial basic potentials $\widetilde{V}_r^{(n+k-n_i)}$, $i = 1,\ldots,k$, $\widetilde{V}_r^{(n+k-1+l)}$ and $\widetilde{V}_r^{(-l)}$, $l = 1, 2, \ldots$ fulfil the following generating equations:

$$\xi^{n+k-1+l} + \widetilde{V}_1^{(n+k-1+l)}\xi^{(n+k)-1} + \cdots + \widetilde{V}_{n+k}^{(n+k-1+l)} = 0, \tag{5.13a}$$

$$\xi^{-l} + \widetilde{V}_1^{(-l)}\xi^{(n+k)-1} + \cdots + \widetilde{V}_{n+k}^{(-l)} = 0, \tag{5.13b}$$

$$\xi^{n+k-n_i} + \widetilde{V}_1^{(n+k-n_i)}\xi^{(n+k)-1} + \cdots + \widetilde{V}_{n+k}^{(n+k-n_i)} = 0, \tag{5.13c}$$

and hence enter the separation curve

$$\widetilde{H}_1\xi^{(n+k)-1} + \widetilde{H}_2\xi^{(n+k)-2} + \cdots + \widetilde{H}_{n+k} = \tfrac{1}{2}f(\xi)\mu^2 + \gamma(\xi), \qquad \widetilde{H}_{n_i} = 0, \tag{5.14}$$

as $\gamma(\xi) = -\xi^{(n+k)-n_i}, -\xi^{n+k-1+l}, -\xi^{-l}$, where $i = 1,\ldots,k$, $l = 1, 2, \ldots$.

Also as in the one-hole case, it is not difficult to show that basic potentials $\widetilde{V}_r^{(m)}$, $m > n+k$ can be constructed recursively by the following recursion relation:

$$\widetilde{V}_r^{(m+1)} = \widetilde{V}_{r+1}^{(m)} - \widetilde{V}_r^{(n+k)}\widetilde{V}_1^{(m)} - \sum_{i=1}^{k}\widetilde{V}_r^{(n+k-n_i)}\widetilde{V}_{n_i+1}^{(m)}, \tag{5.15}$$

where $\widetilde{V}_r^{(n+k-n_i)}$, $i = 1,\ldots,k$ and $\widetilde{V}_r^{(n+k)}$ are given by (5.11) and (5.12).

5.4. Quasi-bi-Hamiltonian representation

Theorem 11. *Hamiltonian functions $\widetilde{H}_r$ belong to the following quasi-bi- Hamiltonian chain:*

$$\begin{gathered} \mathrm{d}\widetilde{H}_{r+1} = N^* \,\mathrm{d}\widetilde{H}_r + \alpha_r^0 \,\mathrm{d}\widetilde{H}_1 + \sum_{i=1}^{k} \alpha_r^{n_i} \,\mathrm{d}\widetilde{H}_{n_i+1} \\ \Updownarrow \\ \pi_0 \,\mathrm{d}\widetilde{H}_{r+1} = \pi_1 \,\mathrm{d}\widetilde{H}_r + \alpha_r^0 \pi_0 \,\mathrm{d}\widetilde{H}_1 + \sum_{i=1}^{k} \alpha_r^{n_i} \pi_0 \,\mathrm{d}\widetilde{H}_{n_i+1}, \end{gathered} \tag{5.16}$$

where $\alpha_r^s = \widetilde{V}_r^{(n+k)-s}$. Of course formula (5.16) works separately for the geodesic Hamiltonians $\widetilde{E}_r$ and potentials $\widetilde{V}_r$.

The proof is inductive and we skip it as it involves too many technicalities.

As in the one-hole case (4.20), the components of $\mathrm{d}p_i$ in (4.19*a*) give us the analogue of formulae (3.2) for the k-hole case

$$\widetilde{K}_{r+1} = L\widetilde{K}_r + \alpha_r^0 I + \sum_{i=1}^{k} \alpha_r^{n_i} \widetilde{K}_{n_i+1}. \tag{5.17}$$

Then, from (5.17) we find that for function $\widetilde{L} = \frac{1}{2} p^T L \widetilde{G} p$ the following relation holds:

$$\widetilde{L} = \widetilde{E}_2 - \alpha_1^0 \widetilde{E}_1 - \sum_{i=1}^{k} \alpha_r^{n_i} \widetilde{E}_{n_i+1}, \tag{5.18}$$

hence

$$\{\widetilde{L}, \widetilde{E}_1\}_{\pi_0} = \left\{\widetilde{E}_1, \alpha_1^0\right\}_{\pi_0} \widetilde{E}_1 + \sum_{i=1}^{k} \left\{\widetilde{E}_1, \alpha_1^{n_i}\right\}_{\pi_0} \widetilde{E}_{n_1+1} = \kappa^0 \widetilde{E}_1 + \sum_{i=1}^{k} \kappa^{n_i} \widetilde{E}_{n_i+1}. \tag{5.19}$$

So obviously, L is not a conformal Killing tensor for $\widetilde{G}$ given by (5.7).

Note, that although the number k can be arbitrary large, nevertheless, the maximal number of nonvanishing terms $\alpha_r^{n_i} d\widetilde{H}_{n_i+1}$ in (5.16) is lower or equal to n. In fact, if n_i and n_{i+1} are two successive numbers, i.e. $n_{i+1} = n_i + 1$, then $\alpha_r^{n_i} \,\mathrm{d}\widetilde{H}_{n_i+1} = \alpha_r^{n_i} \,\mathrm{d}\widetilde{H}_{n_{i+1}} = 0$, as from construction $\widetilde{H}_{n_{i+1}} = 0$. Hence, for a string of successive numbers $n_i - s_i + 1, n_i - s_i + 2, \ldots, n_i$, only the term with n_i is nonzero in formula (5.16), as

$$\widetilde{H}_{n_i - s_i + 1} = \widetilde{H}_{n_i - s_i + 2} = \cdots = \widetilde{H}_{n_i} = 0.$$

Thus, assume that in the sequence

$$\widetilde{H}_1 \xi^{(n+k)-1} + \widetilde{H}_2 \xi^{(n+k)-2} + \cdots + \widetilde{H}_{n+k}$$

we have l strings of holes, where the ith string has s_i holes and $s_1 + \cdots + s_l = k$. Then, the quasi-bi-Hamiltonian chain (5.16) takes the form

$$\mathrm{d}\widetilde{H}_{r+1} = N^* \,\mathrm{d}\widetilde{H}_r + \alpha_r^{n_0} \,\mathrm{d}\widetilde{H}_1 + \alpha_r^{n_1} \,\mathrm{d}\widetilde{H}_{n_1+1} + \cdots + \alpha_r^{n_l} \,\mathrm{d}\widetilde{H}_{n_l+1}, \tag{5.20}$$

where $n_0 = 0$.

Remark 12. The systems considered in this paper, although obtained through the deformation procedure on the level of Hamiltonian functions, are far from being trivial generalizations of Benenti systems. There are no obvious relations between the solutions of a given Benenti system and all its deformations. In each case we have a different inverse Jacobi problem to solve. Note, that the common feature of appropriate deformed systems is the same set of separated coordinates, determined by the related Benenti system.

All systems considered in this paper can be lifted to a pure bi-Hamiltonian form on an extended phase space. For the Benenti class it was done in [26] while for the other classes the lift was constructed in [32].

6. Examples

6.1. Seventh-order stationary KdV

Let us consider the so-called first Newton representation of the seventh-order stationary flow of the KdV hierarchy [33, 18]. It is a Lagrangian system of second-order Newton equations

$$\begin{aligned} q^1_{tt} &= -10(q^1)^2 + 4q^2 \qquad q^2_{tt} = -16q^1q^2 + 10(q^1)^3 + 4q^3 \\ q^3_{tt} &= -20q^1q^3 - 8(q^2)^2 + 30(q^1)^2q^2 - 15(q^1)^4 \end{aligned} \tag{6.1}$$

with the corresponding Lagrangian

$$\mathcal{L} = q^1_t q^3_t + \tfrac{1}{2}\left(q^2_t\right)^2 + 4q^2q^3 - 10(q^1)^2q^3 - 8q^1(q^2)^2 + 10(q^1)^3q^2 - 3(q^1)^5,$$

so that it can be cast in a Hamiltonian form. In fact, the above system is a separable system from the Benenti class, where

$$\begin{aligned} H_1 &= p_1p_3 + \tfrac{1}{2}p_2^2 + 10(q^1)^2q^3 - 4q^2q^3 + 8q^1(q^2)^2 - 10(q^1)^3q^2 + 3(q^1)^5 \\ &= E_1 + V_1(q), \\ H_2 &= \tfrac{1}{2}q^3p_3^2 - \tfrac{1}{2}q^1p_2^2 + \tfrac{1}{2}q^2p_2p_3 - \tfrac{1}{2}p_1p_2 - \tfrac{1}{2}q^1p_1p_3 + 2(q^1)^2(q^2)^2 + \tfrac{5}{2}(q^1)^4q^2 \\ &\quad - \tfrac{5}{4}(q^1)^6 - 2(q^2)^3 + (q^3)^2 - 6q^1q^2q^3 \\ &= E_2 + V_2(q), \\ H_3 &= \tfrac{1}{8}(q^2)^2p_3^2 + \tfrac{1}{8}(q^1)^2p_2^2 + \tfrac{1}{8}p_1^2 + \tfrac{1}{4}q^1p_1p_2 + \tfrac{1}{4}q^2p_1p_3 - \tfrac{1}{4}q^1q^2p_2p_3 \\ &\quad - \tfrac{1}{2}q^3p_2p_3 - 3(q^1)^3(q^2)^2 + q^1(q^2)^3 + \tfrac{5}{4}(q^1)^5q^2 + 2q^1(q^3)^2 \\ &\quad + \tfrac{5}{4}(q^1)^4q^3 + (q^2)^2q^3 - (q^1)^2q^2q^3 \\ &= E_3 + V_3(q), \end{aligned}$$

with the corresponding operators π_0 and π_1

$$\pi_0 = \begin{pmatrix} 0_3 & I_3 \\ -I_3 & 0_3 \end{pmatrix}, \qquad \pi_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & q^1 & -1 & 0 \\ 0 & 0 & 0 & q^2 & 0 & -1 \\ 0 & 0 & 0 & 2q^3 & q^2 & q^1 \\ -q^1 & -q^2 & -2q^3 & 0 & p_2 & p_3 \\ 1 & 0 & -q^2 & -p_2 & 0 & 0 \\ 0 & 1 & -q^1 & -p_3 & 0 & 0 \end{pmatrix}.$$

From the form of H_1 one can directly see that the inverse metric tensor G and the conformal Killing tensor L, expressed in (q, p) variables, have the form

$$G = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad L = \frac{1}{2}\begin{pmatrix} q^1 & -1 & 0 \\ q^2 & 0 & -1 \\ 2q^3 & q^2 & q^1 \end{pmatrix},$$

and hence $K_1 = I$, $A_1 = G$,

$$K_2 = \frac{1}{2}\begin{pmatrix} -q^1 & -1 & 0 \\ q^2 & -2q^1 & -1 \\ 2q^3 & q^2 & -q^1 \end{pmatrix}, \qquad K_3 = \frac{1}{4}\begin{pmatrix} q^2 & q^1 & 1 \\ -q^1q^2 - 2q^3 & (q^1)^2 & q^1 \\ (q^2)^2 & -q^1q^2 - 2q^3 & q^2 \end{pmatrix},$$

$$A_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 & q^1 \\ -1 & -2q^1 & q^2 \\ -q^1 & q^2 & 2q^3 \end{pmatrix}, \qquad A_3 = \frac{1}{4}\begin{pmatrix} 1 & q^1 & q^2 \\ q^1 & (q^1)^2 & -q^1q^2 - 2q^3 \\ q^2 & -q^1q^2 - 2q^3 & (q^2)^2 \end{pmatrix}.$$

The quasi-bi-Hamiltonian chain is given by (3.13) with $r = 1, 2, 3$, where

$$\rho_1 = -q^1, \qquad \rho_2 = \tfrac{1}{4}(q^1)^2 + \tfrac{1}{2}q^2, \qquad \rho_3 = -\tfrac{1}{4}q^1q^2 - \tfrac{1}{4}q^3.$$

The transformation $(\lambda, \mu) \to (q, p)$ is constructed from the relations

$$q^1 = \lambda^1 + \lambda^2 + \lambda^3, \qquad \tfrac{1}{4}(q^1)^2 + \tfrac{1}{2}q^2 = \lambda^1\lambda^2 + \lambda^1\lambda^3 + \lambda^2\lambda^3, \qquad \tfrac{1}{4}q^1q^2 + \tfrac{1}{4}q^3 = \lambda^1\lambda^2\lambda^3,$$

(the explicit formulae are given in [18]) and the separation curve for H_i is

$$H_1\xi^2 + H_2\xi + H_3 = \tfrac{1}{8}\mu^2 + 16\xi^7. \tag{6.2}$$

Two-hole deformation.

There are three admissible cases of two-hole deformation: $(n_1 = 2, n_2 = 4)$, $(n_1 = 2, n_2 = 3)$ and $(n_1 = 3, n_2 = 4)$. Here we present the first case. The deformed Hamiltonians are the following:

$$\begin{aligned} \widetilde{H}_1 &= \frac{1}{\rho_1\rho_2 - \rho_3}H_2 = \frac{1}{2}p^T\widetilde{G}p + \widetilde{V}_1, && \widetilde{H}_2 = 0, \\ \widetilde{H}_3 &= H_1 + \frac{\rho_2 - \rho_1^2}{\rho_1\rho_2 - \rho_3}H_2 = \frac{1}{2}p^T\widetilde{A}_3p + \widetilde{V}_3, && \widetilde{H}_4 = 0, \\ \widetilde{H}_5 &= H_3 + \frac{\rho_1\rho_3}{\rho_1\rho_2 - \rho_3}H_2 = \frac{1}{2}p^T\widetilde{A}_5p + \widetilde{V}_5, \end{aligned} \tag{6.3}$$

where

$$\widetilde{A}_1 = \widetilde{G} = \frac{1}{\rho_1\rho_2 - \rho_3}K_1K_2G = \frac{2}{q_1^3 - q_1q_2 + q_3}\begin{pmatrix} 0 & 1 & q^1 \\ 1 & \frac{1}{2}q^1 & -q^2 \\ q^1 & -q^2 & -\frac{1}{2}q^3 \end{pmatrix},$$

$$\widetilde{A}_2 = 0, \qquad \widetilde{A}_3 = \frac{1}{\rho_1\rho_2 - \rho_3}K_2\big(K_3 - K_2^2\big)G, \qquad \widetilde{A}_4 = 0, \qquad \widetilde{A}_5 = \frac{1}{\rho_1\rho_2 - \rho_3}K_2K_3^2G,$$

$$\widetilde{V}_1 = \frac{4}{q_3 - q_1q_2 - q_1^3}V_2, \qquad \widetilde{V}_2 = 0, \qquad \widetilde{V}_3 = V_1 + \frac{2q_2 - 3q_1^2}{q_3 - q_1q_2 - q_1^3}V_2,$$

$$\widetilde{V}_4 = 0, \qquad \widetilde{V}_5 = V_3 + \frac{q_1(q_1q_2 + q_3)}{q_3 - q_1q_2 - q_1^3}V_2.$$

The quasi-bi-Hamiltonian chain takes the form

$$\mathrm{d}\widetilde{H}_{r+1} = N^*\,\mathrm{d}\widetilde{H}_r + \alpha_r^0\,\mathrm{d}\widetilde{H}_1 + \alpha_r^2\,\mathrm{d}\widetilde{H}_3 + \alpha_r^4\,\mathrm{d}\widetilde{H}_5, \qquad r = 1, \ldots, 5,$$

$$\alpha_r^0 = \rho_r + \frac{\rho_{r-2}\rho_2\rho_3 - \rho_{r-1}\rho_2^2}{\rho_1\rho_2 - \rho_3}, \qquad \alpha_r^2 = \frac{\rho_{r-1}\rho_2 - \rho_{r-2}\rho_3}{\rho_1\rho_2 - \rho_3}, \qquad a_r^4 = \frac{\rho_{r-2}\rho_1 - \rho_{r-1}}{\rho_1\rho_2 - \rho_3}. \tag{6.4}$$

The respective separation curve is

$$\widetilde{H}_1\xi^4 + \widetilde{H}_3\xi^2 + \widetilde{H}_5 = \tfrac{1}{8}\mu^2 + 16\xi^7. \tag{6.5}$$

7. Summary

We have considered a geometric separability theory of Liouville integrable systems with n quadratic in momenta constants of motion

$$H_i(q,p) = \tfrac{1}{2}p^T A_i(q)p + V_i(q), \qquad i = 1,\ldots,n \tag{7.1}$$

and with separation curves of the polynomial type

$$H_1\xi^{m_1} + \cdots + H_n\xi^{m_n} = \tfrac{1}{2}f(\xi)\mu^2 + \gamma(\xi), \qquad m_n = 0 < m_{n-1} < \cdots < m_1 \in \mathbb{N}, \tag{7.2}$$

where $f(\xi)$ and $\gamma(\xi)$ are Laurent polynomials of ξ. The main result of the paper is the systematic construction of new separable Stäckel systems given by Hamiltonian functions (7.1) and separation curves (7.2), by appropriate deformations of Benenti systems characterized by a separation curves in the form

$$H_1\xi^{n-1} + H_2\xi^{n-2} + \cdots + H_n = \tfrac{1}{2}f(\xi)\mu^2 + \gamma(\xi).$$

The most fundamental and surprising result of this paper can be formulated in the following way. If, for a given canonical coordinate system $(q_i, p_i), i = 1,\ldots,n$, we have a pair of objects, i.e. contravariant metric tensor $G^{(0)}$ and related special conformal Killing tensor L, then we can construct systematically, in these coordinates, infinitely many different Liouville integrable and separable Hamiltonian systems (7.1) with respective separation curves of the form (7.2) and with explicit form of transformation to separated coordinates. Observe that according to what was said in section 4, the separation curve of geodesic motion for $G^{(0)}$ is the following:

$$E_1\xi^{n-1} + E_2\xi^{n-2} + \cdots + E_n = \tfrac{1}{2}\mu^2. \tag{7.3}$$

So, the passage from system (7.3) to systems (7.2) is constructive and determined completely by the pair $(G^{(0)}, L)$.

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